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NOTE

AN AXIOMATIZATION OF THE τ-VALUE

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The τ-value is characterized by three axioms. It is shown that the τ-value is the unique solution concept which is efficient and has the minimal right property and the restricted proportionality property. The minimal right property is weaker than the additivity property, which plays a role in the axiomatic characterization of the Shapley value: together with individual rationality and efficiency additivity implies the minimal right property. The restricted proportionality property says that for games with minimal right vector zero, the dividend given to the players is proportional to the marginal contribution of the players to the grand coalition.

Key words: τ-value, minimal right property, Shapley value, cooperative game.

1. Introduction

We consider n-person sidepayment games. The player set $\{1,2,\dots,n\}$ is denoted by N, the set of coalitions by 2^N. Such a game is a real-valued function $v:2^N\to\mathbb{R}$ with $v(\emptyset)=0$, which assigns to each coalition S its worth $v(S)$. The worth $v(S)$ can be interpreted as the reward, which the players in S can obtain by working together. We denote the set of n-person sidepayment games by G^n. The main problem in cooperative game theory is: how to divide $v(N)$ among the players if the grand coalition forms? One can find in the literature many proposals (solution concepts) as answer to this question. See Owen (1982) or Rosenmüller (1981) for surveys. Among the single-valued solution concepts the Shapley value (Shapley, 1952) and the nucleolus (Schmeidler, 1969) have drawn much attention for a long time. For both solution concepts there are known axiomatic characterizations (cf. Shapley, 1953; Sobolev, 1975). For the single-valued τ-value, introduced in 1981 in Tijs (1981) till now there has been no axiomatization. The objective of this paper is to fill this gap.

Let us recall some notions, which play a role in the definition of the τ-value. Let $v\in G^n$. Then the *marginal vector* $M(v)$, corresponding to v, is the vector with ith coordinate

$$M_i(v)=v(N)-v(N-\{i\}).$$

The real number $M_i(v)$ measures the increase in worth, caused, if player i joins the coalition of all the other players. In general, it is reasonable for a player i to expect a dividend of $v(N)$ which does not exceed this marginal contribution $M_i(v)$. If he claims more than $M_i(v)$ it is to the advantage of the coalition $N-\{i\}$ to throw i out of the grand coalition. Hence, $M_i(v)$ can be seen as an upper bound for the dividend or as a *utopia payoff* for player i. With the aid of these utopia payoffs we now derive a lower bound for the dividend which we call the minimal right payoff for the players.

Let us imagine ourselves in the position of player i, who knows the utopia payoffs of the other players and knows also that there are 2^{n-1} possible coalitions to which he belongs. The formation of each of these coalitions is attractive for the other players of the coalition, if player i is willing to pay to those players their utopia payoff. If S is such a coalition, and player i pays the other players this payoff, then for player i is left the payoff

$$R_v(S,i)=v(S)-\sum_{j\in S-\{i\}} M_j(v),$$

which we call the *remainder* for player i in the coalition S. It is clear that $R_v(S,i)$ can be seen as a lower bound for the dividend of player i in the grand coalition. Among the 2^{n-1} possible coalitions player i can choose one with maximal remainder for him. The payoff for player i in such a coalition is

$$m_i(v)=\max_{S\ni i} R_v(S,i)$$

which we call the *minimal right* for player i. The vector

$$m(v)=(m_1(v),m_2(v),\ldots,m_n(v))$$

is called the *minimal right vector*. Now we consider the subclass Q^n of n-person sidepayment games with the properties:

(i) for each player the minimal right does not exceed the utopia payoff,

(ii) the sum of all minimal rights of the players does not exceed $v(N)$,

(iii) the sum of the utopia payoffs of all players is at least $v(N)$.

Hence, $Q^n=\{v\in G^n: m(v)\le M(v),\ \sum_{i=1}^n m_i(v)\le v(N)\le \sum_{i=1}^n M_i(v)\}$. This family Q^n of quasi-balanced games is a full-dimensional cone in the (2^n-1)-dimensional linear space G^n and contains the family of balanced games as a subset (cf. Tijs, 1981). The τ-value is a solution concept, defined on Q^n, assigning to each $v\in Q^n$ a unique vector $\tau(v)\in\mathbb{R}^n$, which is a compromise between the minimal right vector and the utopia vector of the game. For $v\in Q^n$, $\tau(v)$ is by definition the unique vector lying on the line through $m(v)$ and $M(v)$ which is efficient, i.e. which lies in the hyperplane

$$H=\left\{x\in\mathbb{R}^n: \sum_{i=1}^n x_i=v(N)\right\}.$$

(see Fig. 1). In formula,

$$\tau(v) = m(v) + \alpha_v (M(v) - m(v)) \tag{1}$$

where $\alpha_v = 0$ if $M(v) = m(v)$ and, otherwise,

$$\alpha_v = \left(\sum_{i=1}^{n} M_i(v) - \sum_{i=1}^{n} m_i(v) \right)^{-1} \left(v(N) - \sum_{i=1}^{n} m_i(v) \right). \tag{2}$$

In the next section we characterize the map $\tau : Q^n \to \mathbb{R}^n$ with the aid of three properties. We conclude this section by mentioning the monograph of Driessen (1985), which gives a thorough treatment of many of the known results for the τ-value.

2. An axiomatic characterization of the τ-value

Let $f : Q^n \to \mathbb{R}^n$ be a map. Then f is called an *efficient rule* if

$$\sum_{i=1}^{n} f_i(v) = v(N) \quad \text{for all } v \in Q^n.$$

Furthermore, f is said to have the *minimal right property* if

$$f(v) = m(v) + f(v - m(v)) \quad \text{for all } v \in Q^n. \tag{3}$$

Here $v - m(v)$ is called the *right reduced game*, corresponding to v, which is obtained by subtracting from v the additive game corresponding to the minimal right vector of v. Hence, $(v - m(v))(S) = v(S) - \sum_{i \in S} m_i(v)$ for all $S \in 2^N$. Note that $v - m(v) \in Q^n$ if $v \in Q^n$.

For a rule f with the minimal right property it does not matter for a player i, whether he gets the dividend $f_i(v)$ which is described by the rule f in the game v, or whether he obtains first his minimal right payoff in v and then also the dividend which the rule f assigns to him in the right reduced game $v - m(v)$.

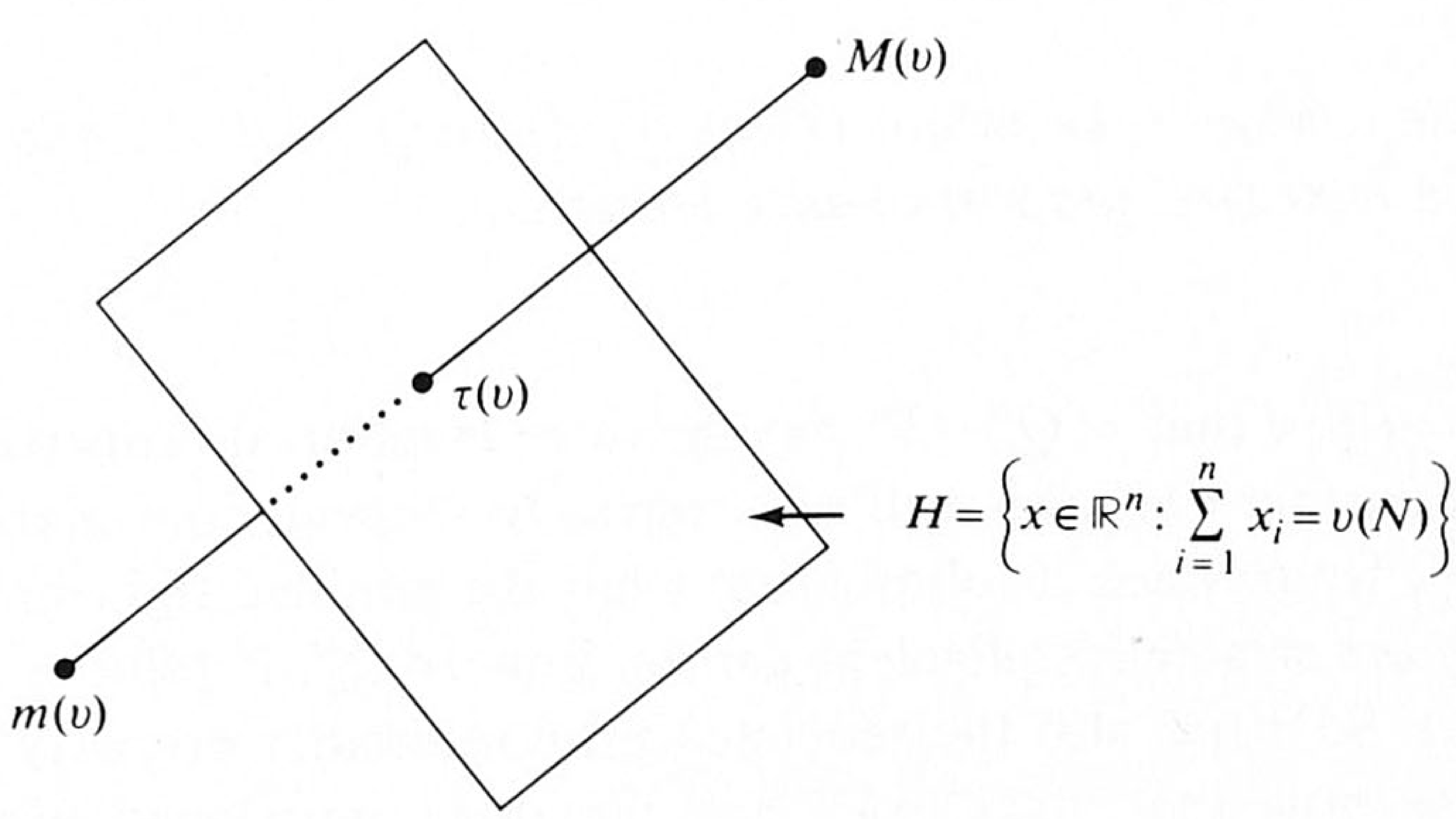

Fig. 1.

Note that (3) is a weak form of additivity: an additive rule, where

$$f(v+w)=f(v)+f(w) \quad \text{for all } v, w \in Q^n$$

which assigns to additive games the unique efficient and individual payoff vector automatically has the minimal right property.

Let us denote by Q_0^n the family of quasi-balanced games with minimal right vector zero. So,

$$Q_0^n := \{v \in Q^n : m(v)=0\}.$$

Note that

$$v - m(v) \in Q_0^n \quad \text{if } v \in Q^n. \tag{4}$$

This follows by noting that for $w := v - m(v)$ we have:

$$\begin{aligned} R_w(S,i) &= w(S) - \sum_{j \in S-\{i\}} M_j(w) \\ &= \Big(v(S) - \sum_{j \in S} m_j(v)\Big) - \Big(\sum_{j \in S-\{i\}} M_j(v) - \sum_{j \in S-\{i\}} m_j(v)\Big) \\ &= R_v(S,i) - m_i(v), \end{aligned}$$

and then

$$\max_{S \ni i} R_w(S,i) = \max_{S \ni i} R_v(S,i) - m_i(v) = 0.$$

Let us say that a rule $f: Q^n \to \mathbb{R}^n$ has the *restricted proportionality property* if for each $v \in Q_0^n$:

$f(v)$ is a multiple of the utopia vector $M(v)$.

Hence, for games with minimal rights zero the dividend of the players is proportional to the marginal contributions of the players to the grand coalition.

Our main result can be stated now.

Theorem. *The τ-value is the unique efficient rule on Q^n with the minimal right property and the restricted proportionality property.*

Proof.

(i) First we show that $\tau: Q^n \to \mathbb{R}^n$ has the three mentioned properties. Obviously, τ is efficient. Further τ behaves well with respect to strategic equivalence as is proved in Tijs (1981), from which it follows that τ has the minimal right property, since v and $v - m(v)$ are strategic equivalent games. For $v \in Q_0^n$, it follows from (1), that $\tau(v) = \alpha_v M(v)$. So τ has also the restricted proportionality property.

(ii) Suppose now that $f: Q^n \to \mathbb{R}^n$ has the three mentioned properties. Take $v \in Q^n$. We have to show that $f(v) = \tau(v)$. By (4), $v - m(v) \in Q_0^n$. From the restricted proportionality property it follows that there is an $\alpha \in \mathbb{R}$ such that

$$f(v-m(v)) = \alpha M(v-m(v)) = \alpha(M(v)-m(v)). \tag{5}$$

From the minimal right property it follows that

$$f(v) = m(v) + f(v-m(v)). \tag{6}$$

Combining (5) and (6), we obtain

$$f(v) = m(v) + \alpha(M(v)-m(v)).$$

The efficiency of f then implies that, in case $M(v) \neq m(v)$, α is equal to α_v as defined in (2). But then $f(v) = \tau(v)$ in view of (1) and (2). □

We conclude this paper by noting that a list of properties of the τ-value can be found in Tijs (1981) and also two other properties in Tijs and Driessen (1986).

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